INTEGRAL COMPLETE r-PARTITE GRAPHS $K_{m,m,p}$ AND $K_{m,m,m,p}$

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Abstract. A graph is called integral if all the eigenvalues of its adjacency matrix are integers. In this paper, we give sufficient conditions for complete 3-partite graphs $K_{m,m,p}$ and complete 4-partite graphs $K_{m,m,m,p}$ to be integral, from which we construct infinitely many new classes of such integral graphs.

Key words: Integral Graph, Complete 3-partite Graph, Complete 4-partite Graph, Divisor, Characteristic Polynomial, Graph Spectrum, Diophantine Equation.

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1. Introduction

We shall consider only simple undirected graphs. For a graph G, let $V(G)$ denote the vertex set and $E(G)$ the edge set. The characteristic polynomial $|x \cdot I - A|$ of the adjacency matrix A of G is called the characteristic polynomial of G and is denoted by $P(G,x)$. The spectrum of A is also called the spectrum of G.

A complete 3-partite graph $K_{m,n,p}$ is a graph with a vertex set $V = V_1 \cup V_2 \cup V_3$, where $|V_1| = m$, $|V_2| = n$, $|V_3| = p$, such that two vertices in V are adjacent if and only if they belong to the different V_i's. A complete 4-partite graph $K_{m,n,p,q}$ is defined analogously.

The notion of integral graphs was introduced by F. Harary and A. J. Schwenk in 1974 (see [4]). A graph G is called integral if all the zeros of the characteristic polynomial $P(G,x)$ are integers. In general, the problem of characterizing integral graphs seems to be very difficult. Thus, it makes sense to restrict our investigations to some families of graphs, for example cubic graphs, trees, complete n-partite graphs, graphs with three eigenvalues, graphs with maximum degree 4, etc. The results on integral graphs can be found in [1, 2, 3, 4, 5, 11]. Integral complete 3-partite graphs have been studied in [8, 10]. Integral complete r-partite graphs have been studied in [9, 12, 13, 14]. In [12] some sufficient and necessary conditions for complete r-partite graphs to be integral were given, from which new classes of integral 3-partite graphs were constructed. In [12] several open problems for integral complete r-partite graphs were given. Finally, in [7] a class of integral complete 4-partite graphs were constructed. In this paper, we give sufficient conditions for complete 3-partite graphs $K_{m,m,p}$ and complete 4-partite graphs $K_{m,m,m,p}$ to be integral, from which we construct infinitely many new classes of such integral graphs.

For all other facts on terminology of graph spectra see [2, 3].

2. Preliminaries

The following theorem gives sufficient conditions for the first known infinite class of integral complete tripartite graphs.

Theorem 1. (Roitman [10]) *Let* $m = 4u^2\left(u^2+v^2\right)^3$, $n = 3u^2v^2\left(u^2+6uv+v^2\right)\left(-u^2+6uv-v^2\right)$, $p = 4v^2\left(u^2+v^2\right)^3$ *such that* $\left(3-\sqrt{8}\right)v < u < v$ *and let* $x_1 = 24u^2v^2\left(u^2+v^2\right)^2$, $x_2 = -2uv\left(u^2+v^2\right)^2\left(-u^2+6uv-v^2\right)$, $x_3 = -2uv\left(u^2+v^2\right)^2\left(u^2+6uv+v^2\right)$ *such that* u,v *are positive integers. Then* $K_{m,n,p}$ *is integral.*

Another sufficient conditions for integral complete 3-partite graphs were given in [8, 12] and are summarized in the following theorems.

Theorem 2. (Híc, Pokorný, Černek [8]) *Let* $m = \dfrac{s^2\left(s^2-t^2\right)}{dk^2}, n = \dfrac{2t^2s^2}{dk^2}, p = \dfrac{t^2\left(9t^2-s^2\right)}{dk^2}$, *where* $t < s < 3t, dk^2 = \left(s^2\left(s^2-t^2\right), 2t^2s^2, t^2\left(9t^2-s^2\right)\right)$. Then $K_{m,n,p}$ *is integral. The zeros of its divisor are* $x_1 = \dfrac{4t^2s^2}{dk^2}, x_2 = \dfrac{ts\left(s-3t\right)\left(t+s\right)}{dk^2}, x_3 = \dfrac{ts\left(s+3t\right)\left(t-s\right)}{dk^2}$.

Theorem 3. (Wang, Li, Hoede [12]) *Let* $m = \dfrac{a^2+b^2}{d}, n = \dfrac{2a(2a+b)}{d}, p = \dfrac{2b(2b-a)}{d}$ *and* $x_1 = \dfrac{2b(b+2a)}{d}, x_2 = -\dfrac{2a(2b-a)}{d}, x_3 = -\dfrac{2(a^2+b^2)}{d}$, *where* $a > 0, b > 0$, $d = \left(2b\left(2b-a\right), 2a\left(2a+b\right), a^2+b^2\right)$. *Then* $K_{m,n,p}$ *is integral.*

In [12] the folloving theorem is given.

Theorem 4. (Wang, Li, Hoede [12]) *For any positive integer* t, *the complete* r*-partite graph* $K_{p_1\cdot t, p_2\cdot t, \dots, p_r\cdot t}$ *is integral if and only if the complete* r*-partite graph* $K_{p_1, p_2, \dots, p_r}$ *is integral.*

Remark. The above theorem shows that it is reasonable to study only complete r-partite graphs with $(p_1, p_2, \dots, p_r) = 1$. Let us call such a complete graph primitive. So, in general, the primitive complete r-partite graphs are the only ones which are of interest.

Corollary 5. *For any positive integer* t, *the complete 3-partite graph* $K_{m,n,p}$ *is integral if and only if the complete 3-partite graph* $K_{m\cdot t, n\cdot t, p\cdot t}$ *is integral.*

Corollary 6. *For any positive integer* t, *the complete 4-partite graph* $K_{m,n,p,r}$ *is integral if and only if the complete 4-partite graph* $K_{m\cdot t, n\cdot t, p\cdot t, r\cdot t}$ *is integral.*

In above theorems we consider a case $m < n < p$.
It is easy to prove that the following theorems holds.

Theorem 7. *The graph* $K_{p,p,p}$ *is integral for every* $p \in N$ *and its spectrum is* $\left\{-p,-p,2p,\underbrace{0,...,0}_{3p-3}\right\}$.

Theorem 8. *The graph* $K_{p,p,p,p}$ *is integral for every* $p \in N$ *and its spectrum is* $\left\{-p,-p,-p,3p,\underbrace{0,...,0}_{4p-4}\right\}$.

3. Results

In the following part of the paper we explore complete 3-partite graphs $K_{m,m,p}$. We give sufficient conditions for the graph $K_{m,m,p}$ to be integral.

From the theory of divisors and co-divisors of a graph follows that the divisor of the graph $K_{m,m,p}$ has characteristic polynomial

(1) $$P(D,x) = x^3 - (m^2 + 2mp)x - 2m^2 p.$$

Moreover, the characteristic polynomial of the co-divisor is

(2) $$P(C,x) = x^{2m+p-3}.$$

More details about the theory of divisors and co-divisors can be found in [2, 5, 6] . That the following theorem holds.

Theorem 9. *Let* k *be a nonnegative integer. Let* n *be a positive integer. Let* $m = (2k+1)^2$, $p = \sum_{i=1}^{n}(i+k) = \frac{n^2+n}{2} + nk$. *Then the complete 3-partite graph* $K_{m,m,p}$ *is integral.*

Proof:
From (2) it is clear that the zeros of $P(C,x)$ are only 0. As $P(G,x) = P(D,x)\cdot P(C,x)$, the graph $K_{m,m,p}$ is integral if and only if its divisor $P(D,x)$ has only integer zeros.
The divisor of the graph $K_{m,m,p}$ is $x^3 - (m^2+2mp)x - 2m^2p = 0$.
It is easy to prove that $x_1 = -m$ is a zero of $x^3 - (m^2+2mp)x - 2m^2p = 0$. Then $x^3 - (m^2+2mp)x - 2m^2p = \left(x^2 - mx - 2mp\right)(x+m)$.
Using $m = (2k+1)^2$ and $p = \frac{n^2+n}{2} + nk$ to $x^2 - mx - 2mp = 0$ we get the quadratic equation $x^2 - (2k+1)^2 x - 2(2k+1)^2\left(\frac{n^2+n}{2} + nk\right) = 0$. After some routine calculations it is easy to prove that its zeros are $x_2 = -n(2k+1)$ and $x_3 = (2k+1)(2k+n+1)$. □

Some of complete 3-partite graphs constructed by Theorem 9 are in Table 1.

k	n	m	p	zeros of the divisor		
0	1	1	1	-1	-1	2
0	2	1	3	-1	-2	3
0	3	1	6	-1	-3	4
0	4	1	10	-1	-4	5
0	5	1	15	-1	-5	6
1	1	9	2	-9	-3	12

1	2	9	5	-9	-6	15
1	3	9	9	-9	-9	18
1	4	9	14	-9	-12	21
1	5	9	20	-9	-15	24
2	1	25	3	-25	-5	30
2	2	25	7	-25	-10	35
2	3	25	12	-25	-15	40
2	4	25	18	-25	-20	45
2	5	25	25	-25	-25	50
3	1	49	4	-49	-7	56
3	2	49	9	-49	-14	63
3	3	49	15	-49	-21	70
3	4	49	22	-49	-28	77
3	5	49	30	-49	-35	84

Table 1

Theorem 10. *Let k be a positive integer. Let $n > k$ be a positive integer. Let $m = 8k^2$, $p = n^2 - k^2$. Then the complete 3-partite graph $K_{m,m,p}$ is integral.*

Proof:

Similarly to theorem 9, it is sufficient to prove that the divisor $P(D,x) = (x^2 - mx - 2mp)(x + m)$ of the graph $K_{m,m,p}$ has only integer zeros.

Using $m = 8k^2$ and $p = n^2 - k^2$ to $x^2 - mx - 2mp = 0$ we get the quadratic equation $x^2 - 8k^2x - 16k^2(n^2 - k^2) = 0$. After some routine calculations it is easy to prove that its zeros are $x_2 = 4k(k - n)$ and $x_3 = 4k(k + n)$. □

Some of complete 3-partite graphs constructed by Theorem 10 are in Table 2.

k	*n*	*m*	*p*	zeros of the divisor		
1	2	8	3	-8	-4	12
1	3	8	8	-8	-8	16
1	4	8	15	-8	-12	20
1	5	8	24	-8	-16	24
1	6	8	35	-8	-20	28
2	3	32	5	-32	-8	40
2	4	32	12	-32	-16	48
2	5	32	21	-32	-24	56
2	6	32	32	-32	-32	64
2	7	32	45	-32	-40	72
3	4	72	7	-72	-12	84
3	5	72	16	-72	-24	96
3	6	72	27	-72	-36	108
3	7	72	40	-72	-48	120
3	8	72	55	-72	-60	132
4	5	128	9	-128	-16	144
4	6	128	20	-128	-32	160

4	7	128	33	-128	-48	176
4	8	128	48	-128	-64	192
4	9	128	65	-128	-80	208

Table 2

Using Theorem 9, Theorem 10 and Theorem 4 we get the following corollary.

Corollary 11.
1. Let k be a nonnegative integer. Let n and t be positive integers. Let $m=(2k+1)^2$, $p=\sum_{i=1}^{n}(i+k)=\frac{n^2+n}{2}+nk$. Then the complete 3-partite graph $K_{m\cdot t,m\cdot t,p\cdot t}$ is integral.
2. Let k and t be positive integers. Let $n>k$ be a positive integer. Let $m=8k^2$, $p=n^2-k^2$. Then the complete 3-partite graph $K_{m\cdot t,m\cdot t,p\cdot t}$ is integral.

Using computer we constructed the list of integral complete 3-partite graphs $K_{m,m,p}$ for $1\le m\le 10000$ and $1\le p\le 10000$. The list contains 57144 integral complete 3-partite graphs. All these graphs can be constructed by Corollary 11.

In [7] the first integral complete 4-partite graphs $K_{m,n,p,q}$ where $m<n<p<q$ were found. In the following part of the paper we explore complete 4-partite graphs $K_{m,m,m,p}$. We give sufficient conditions for the graph $K_{m,m,m,p}$ to be integral.

From the theory of divisors and co-divisors of a graph follows that the divisor of the graph $K_{m,m,m,p}$ has characteristic polynomial

(3) $$P(D,x)=x^4-3m(m+p)x^2-2m^2(m+3p)x-3m^3p.$$

Moreover, the characteristic polynomial of the co-divisor is

(4) $$P(C,x)=x^{3m+p-4}.$$

The following theorems hold.

Theorem 12. *Let k be a nonnegative integer. Let $n\ge 2k+2$ be a positive integer. Let $m=3(2k+1)^2$, $p=n^2-(2k+1)^2$. Then the complete 4-partite graph $K_{m,m,m,p}$ is integral.*

Proof:
It is sufficient to prove that the divisor of the graph $K_{m,m,m,p}$ has only integer zeros.
The divisor of the graph $K_{m,m,m,p}$ is $x^4-3m(m+p)x^2-2m^2(m+3p)x-3m^3p$.
It is easy to show that $x_1=x_2=-m$ are zeros of $x^4-3m(m+p)x^2-2m^2(m+3p)x-3m^3p=0$. Then $x^4-3m(m+p)x^2-2m^2(m+3p)x-3m^3p=(x^2-2mx-3mp)(x+m)^2$.
Using $m=3(2k+1)^2$ and $p=n^2-(2k+1)^2$ to $x^2-2mx-3mp=0$ we get the quadratic equation $x^2-6(2k+1)^2x+9(2k+1)^2(4k^2+4k-n^2+1)=0$. After some routine calculations it is easy to prove that its zeros are $x_3=3(2k+1)(2k-n+1)$ and $x_4=3(2k+1)(2k+n+1)$. □

Some of complete 4-partite graphs constructed by Theorem 12 are in Table 3.

k	n	m	p	zeros of the divisor			
0	2	3	3	-3	-3	-3	9
0	3	3	8	-3	-3	-6	12
0	4	3	15	-3	-3	-9	15
0	5	3	24	-3	-3	-12	18
1	4	27	7	-27	-27	-9	63
1	5	27	16	-27	-27	-18	72
1	6	27	27	-27	-27	-27	81
1	7	27	40	-27	-27	-36	90
2	6	75	11	-75	-75	-15	165
2	7	75	24	-75	-75	-30	180
2	8	75	39	-75	-75	-45	195
2	9	75	56	-75	-75	-60	210
3	8	147	15	-147	-147	-21	315
3	9	147	32	-147	-147	-42	336
3	10	147	51	-147	-147	-63	357
3	11	147	72	-147	-147	-84	378

Table 3

Theorem 13. *Let k be a positive integer. Let $n \geq 2k+1$ be a positive integer. Let $m = 3(2k)^2$, $p = n^2 - (2k)^2$. Then the complete 4-partite graph $K_{m,m,m,p}$ is integral.*

Proof:

Similarly to theorem 12, it is sufficient to prove that the divisor $P(D,x) = (x^2 - 2mx - 3mp)(x+m)^2$ of the graph $K_{m,m,m,p}$ has only integer zeros.

Using $m = 3(2k)^2$ and $p = n^2 - (2k)^2$ to $x^2 - 2mx - 3mp = 0$ we get the quadratic equation $x^2 - 24k^2x + 36k^2(4k^2 - n^2) = 0$. After some routine calculations it is easy to prove that its zeros are $x_3 = 6k(2k-n)$ and $x_4 = 6k(2k+n)$. □

Some of complete 4-partite graphs constructed by Theorem 13 are in Table 4.

k	n	m	p	zeros of the divisor			
1	3	12	5	-12	-12	-6	30
1	4	12	12	-12	-12	-12	36
1	5	12	21	-12	-12	-18	42
1	6	12	32	-12	-12	-24	48
2	5	48	9	-48	-48	-12	108
2	6	48	20	-48	-48	-24	120
2	7	48	33	-48	-48	-36	132
2	8	48	48	-48	-48	-48	144
3	7	108	13	-108	-108	-18	234
3	8	108	28	-108	-108	-36	252
3	9	108	45	-108	-108	-54	270
3	10	108	64	-108	-108	-72	288

4	9	192	17	-192	-192	-24	408
4	10	192	36	-192	-192	-48	432
4	11	192	57	-192	-192	-72	456
4	12	192	80	-192	-192	-96	480

Table 4

Theorem 14. *Let* k *be a nonnegative integer. Let* $n \geq 2k$ *be a nonnegative integer. Let* $m = (6k+1)^2$, $p = 3n^2 + 4n - 12k^2 - 4k + 1$. *Then the complete 4-partite graph* $K_{m,m,m,p}$ *is integral.*

Proof:

Since the beginning of the proof is similar to proofs of theorems 12 and 13, in proofs of the following theorems we make only the last part.

Using $m = (6k+1)^2$ and $p = 3n^2 + 4n - 12k^2 - 4k + 1$ to $x^2 - 2mx - 3mp = 0$ we get the quadratic equation $x^2 - 2(6k+1)^2 x + 3(6k+1)^2 (12k^2 + 4k - 3n^2 - 4n - 1) = 0$. After some routine calculations it is easy to prove that its zeros are $x_3 = (6k+1)(6k-3n-1)$ and $x_4 = 3(2k+n+1)(6k+1)$. □

Some of complete 4-partite graphs constructed by Theorem 14 are in Table 5.

k	*n*	*m*	*p*	zeros of the divisor			
0	0	1	1	-1	-1	-1	3
0	1	1	8	-1	-1	-4	6
0	2	1	21	-1	-1	-7	9
1	2	49	5	-49	-49	-7	105
1	3	49	24	-49	-49	-28	126
1	4	49	49	-49	-49	-49	147
2	4	169	9	-169	-169	-13	351
2	5	169	40	-169	-169	-52	390
2	6	169	77	-169	-169	-91	429
3	6	361	13	-361	-361	-19	741
3	7	361	56	-361	-361	-76	798
3	8	361	105	-361	-361	-133	855

Table 5

Theorem 15. *Let* k *be a nonnegative integer. Let* $n \geq 2k$ *be a nonnegative integer. Let* $m = (6k+1)^2$, $p = 3n^2 + 8n - 12k^2 - 4k + 5$. *Then the complete 4-partite graph* $K_{m,m,m,p}$ *is integral.*

Proof:

Using $m = (6k+1)^2$ and $p = 3n^2 + 8n - 12k^2 - 4k + 5$ to $x^2 - 2mx - 3mp = 0$ we get the quadratic equation $x^2 - 2(6k+1)^2 x + 3(6k+1)^2 (12k^2 + 4k - 3n^2 - 8n - 5) = 0$. After some routine calculations it is easy to prove that its zeros are $x_3 = 3(6k+1)(2k-n-1)$ and $x_4 = (6k+1)(6k+3n+5)$. □

Some of complete 4-partite graphs constructed by Theorem 15 are in Table 6.

k	n	m	p	zeros of the divisor			
0	0	1	5	-1	-1	-3	5
0	1	1	16	-1	-1	-6	8
0	2	1	33	-1	-1	-9	11
1	2	49	17	-49	-49	-21	119
1	3	49	40	-49	-49	-42	140
1	4	49	69	-49	-49	-63	161
2	4	169	29	-169	-169	-39	377
2	5	169	64	-169	-169	-78	416
2	6	169	105	-169	-169	-117	455
3	6	361	41	-361	-361	-57	779
3	7	361	88	-361	-361	-114	836
3	8	361	141	-361	-361	-171	893

Table 6

Theorem 16. *Let* k *be a nonnegative integer. Let* $n > 2k$ *be a positive integer. Let* $m = (6k+2)^2$, $p = 3n^2 + 4n - 12k^2 - 8k$. *Then the complete 4-partite graph* $K_{m,m,m,p}$ *is integral.*

Proof:

Using $m = (6k+2)^2$ and $p = 3n^2 + 4n - 12k^2 - 8k$ to $x^2 - 2mx - 3mp = 0$ we get the quadratic equation $x^2 - 2(6k+2)^2 x + 3(6k+2)^2 (12k^2 + 8k - 3n^2 - 4n) = 0$. After some routine calculations it is easy to prove that its zeros are $x_3 = 3(6k+2)(2k-n)$ and $x_4 = (6k+3n+4)(6k+2)$. □

Some of complete 4-partite graphs constructed by Theorem 16 are in Table 7.

k	n	m	p	zeros of the divisor			
0	1	4	7	-4	-4	-6	14
0	2	4	20	-4	-4	-12	20
0	3	4	39	-4	-4	-18	26
1	3	64	19	-64	-64	-24	152
1	4	64	44	-64	-64	-48	176
1	5	64	75	-64	-64	-72	200
2	5	196	31	-196	-196	-42	434
2	6	196	68	-196	-196	-84	476
2	7	196	111	-196	-196	-126	518
3	7	400	43	-400	-400	-60	860
3	8	400	92	-400	-400	-120	920
3	9	400	147	-400	-400	-180	980

Table 7

Theorem 17. *Let* k *be a nonnegative integer. Let* $n \geq 2k$ *be a nonnegative integer. Let* $m = (6k+2)^2$, $p = 3n^2 + 8n - 12k^2 - 8k + 4$. *Then the complete 4-partite graph* $K_{m,m,m,p}$ *is integral.*
Proof:
Using $m = (6k+2)^2$ and $p = 3n^2 + 8n - 12k^2 - 8k + 4$ to $x^2 - 2mx - 3mp = 0$ we get the quadratic equation $x^2 - 2(6k+2)^2 x + 3(6k+2)^2 (12k^2 + 8k - 3n^2 - 8n - 4) = 0$. After some routine calculations it is easy to prove that its zeros are $x_3 = (6k+2)(6k-3n-2)$ and $x_4 = (6k+3n+6)(6k+2)$. □

Some of complete 4-partite graphs constructed by Theorem 17 are in Table 8.

k	n	m	p	zeros of the divisor			
0	0	4	4	-4	-4	-4	12
0	1	4	15	-4	-4	-10	18
0	2	4	32	-4	-4	-16	24
1	2	64	12	-64	-64	-16	144
1	3	64	35	-64	-64	-40	168
1	4	64	64	-64	-64	-64	192
2	4	196	20	-196	-196	-28	420
2	5	196	55	-196	-196	-70	462
2	6	196	96	-196	-196	-112	504
3	6	400	28	-400	-400	-40	840
3	7	400	75	-400	-400	-100	900
3	8	400	128	-400	-400	-160	960

Table 8

Theorem 18. *Let* k *be a nonnegative integer. Let* $n > 2k$ *be a positive integer. Let* $m = (6k+4)^2$, $p = 3n^2 + 4n - 12k^2 - 16k - 4$. *Then the complete 4-partite graph* $K_{m,m,m,p}$ *is integral.*
Proof:
Using $m = (6k+4)^2$ and $p = 3n^2 + 4n - 12k^2 - 16k - 4$ to $x^2 - 2mx - 3mp = 0$ we get the quadratic equation $x^2 - 2(6k+4)^2 x + 3(6k+4)^2 (12k^2 + 16k - 3n^2 - 4n + 4) = 0$. After some routine calculations it is easy to prove that its zeros are $x_3 = (6k+4)(6k-3n+2)$ and $x_4 = (6k+3n+6)(6k+4)$.

Some of complete 4-partite graphs constructed by Theorem 18 are in Table 9.

k	n	m	p	zeros of the divisor			
0	1	16	3	-16	-16	-4	36
0	2	16	16	-16	-16	-16	48
0	3	16	35	-16	-16	-28	60
1	3	100	7	-100	-100	-10	210
1	4	100	32	-100	-100	-40	240
1	5	100	63	-100	-100	-70	270
2	5	256	11	-256	-256	-16	528
2	6	256	48	-256	-256	-64	576
2	7	256	91	-256	-256	-112	624

3	7	484	15	-484	-484	-22	990
3	8	484	64	-484	-484	-88	1056
3	9	484	119	-484	-484	-154	1122

Table 9

Theorem 19. *Let* k *be a nonnegative integer. Let* $n > 2k$ *be a positive integer. Let* $m = (6k+4)^2$, $p = 3n^2 + 8n - 12k^2 - 16k$. *Then the complete 4-partite graph* $K_{m,m,m,p}$ *is integral.*

Proof:

Using $m = (6k+4)^2$ and $p = 3n^2 + 8n - 12k^2 - 16k$ to $x^2 - 2mx - 3mp = 0$ we get the quadratic equation $x^2 - 2(6k+4)^2 x + 3(6k+4)^2 (12k^2 + 16k - 3n^2 - 8n) = 0$. After some routine calculations it is easy to prove that its zeros are $x_3 = (6k+4)(6k-3n)$ and $x_4 = (6k+3n+8)(6k+4)$.

Some of complete 4-partite graphs constructed by Theorem 19 are in Table 10.

k	n	m	p	zeros of the divisor			
0	1	16	11	-16	-16	-12	44
0	2	16	28	-16	-16	-24	56
0	3	16	51	-16	-16	-36	68
1	3	100	23	-100	-100	-30	230
1	4	100	52	-100	-100	-60	260
1	5	100	87	-100	-100	-90	290
2	5	256	35	-256	-256	-48	560
2	6	256	76	-256	-256	-96	608
2	7	256	123	-256	-256	-144	656
3	7	484	47	-484	-484	-66	1034
3	8	484	100	-484	-484	-132	1100
3	9	484	159	-484	-484	-198	1166

Table 10

Theorem 20. *Let* k *be a nonnegative integer. Let* $n > 2k+1$ *be a positive integer. Let* $m = (6k+5)^2$, $p = 3n^2 + 4n - 12k^2 - 20k - 7$. *Then the complete 4-partite graph* $K_{m,m,m,p}$ *is integral.*

Proof:

Using $m = (6k+5)^2$ and $p = 3n^2 + 4n - 12k^2 - 20k - 7$ to $x^2 - 2mx - 3mp = 0$ we get the quadratic equation $x^2 - 2(6k+5)^2 x + 3(6k+5)^2 (12k^2 + 20k - 3n^2 - 4n + 7) = 0$. After some routine calculations it is easy to prove that its zeros are $x_3 = (6k+5)(6k-3n+3)$ and $x_4 = (6k+3n+7)(6k+5)$.

Some of complete 4-partite graphs constructed by Theorem 20 are in Table 11.

k	n	m	p	zeros of the divisor			
0	2	25	13	-25	-25	-15	65
0	3	25	32	-25	-25	-30	80
0	4	25	57	-25	-25	-45	95

1	4	121	25	-121	-121	-33	275
1	5	121	56	-121	-121	-66	308
1	6	121	93	-121	-121	-99	341
2	6	289	37	-289	-289	-51	629
2	7	289	80	-289	-289	-102	680
2	8	289	129	-289	-289	-153	731
3	8	529	49	-529	-529	-69	1127
3	9	529	104	-529	-529	-138	1196
3	10	529	165	-529	-529	-207	1265

Table 11

Theorem 21. *Let* k *be a nonnegative integer. Let* $n > 2k$ *be a positive integer. Let* $m = (6k+5)^2$, $p = 3n^2 + 8n - 12k^2 - 20k - 3$. *Then the complete 4-partite graph* $K_{m,m,m,p}$ *is integral.*
Proof:
Using $m = (6k+5)^2$ and $p = 3n^2 + 8n - 12k^2 - 20k - 3$ to $x^2 - 2mx - 3mp = 0$ we get the quadratic equation $x^2 - 2(6k+5)^2 x + 3(6k+5)^2 (12k^2 + 20k - 3n^2 - 8n + 3) = 0$. After some routine calculations it is easy to prove that its zeros are $x_3 = (6k+5)(6k-3n+1)$ and $x_4 = (6k+3n+9)(6k+5)$.

Some of complete 4-partite graphs constructed by Theorem 21 are in Table 12.

k	*n*	*m*	*p*	zeros of the divisor			
0	1	25	8	-25	-25	-10	60
0	2	25	25	-25	-25	-25	75
0	3	25	48	-25	-25	-40	90
1	3	121	16	-121	-121	-22	264
1	4	121	45	-121	-121	-55	297
1	5	121	80	-121	-121	-88	330
2	5	289	24	-289	-289	-34	612
2	6	289	65	-289	-289	-85	663
2	7	289	112	-289	-289	-136	714
3	7	529	32	-529	-529	-46	1104
3	8	529	85	-529	-529	-115	1173
3	9	529	144	-529	-529	-184	1242

Table 12

Using Theorems 12-21 and Theorem 4 we get the following corollary.

Corollary 22.
Let $K_{m,m,m,p}$ be integral complete 4-partite graph constructed by one of theorems 12-21. Let t be a positive integer. Then $K_{m\cdot t, m\cdot t, m\cdot t, p\cdot t}$ is integral complete 4-partite graph.

Using computer we constructed the list of integral complete 4-partite graphs $K_{m,m,m,p}$ for $1 \le m \le 10000$ and $1 \le p \le 10000$. The list contains 50261 integral complete 4-partite graphs. All these graphs can be constructed by Corollary 22.

We also explored complete 4-partite graphs $K_{m,m,p,p}$. The divisor of these graphs is $P(D,x) = x^4 - (m^2 + 4mp + p^2)x^2 - 4mp(m+p)x - 3m^2p^2$. It is easy to show that its zeros are $x_1 = -m, x_2 = -p$.
Then $x^4 - (m^2 + 4mp + p^2)x^2 - 4mp(m+p)x - 3m^2p^2 = (x-m)(x-p)(x^2 - mx - px - 3mp)$. As the zeros of $x^2 - mx - px - 3mp = 0$ are $\frac{m+p \pm \sqrt{m^2 + 14mp + p^2}}{2}$, it is clear that if $K_{m,m,p,p}$ is integral, then $m^2 + 14mp + p^2$ is a perfect square. The following theorem holds.

Theorem 23. *Let $K_{m,m,p,p}$ be an integral complete 4-partite graph. Then $m^2 + 14mp + p^2$ is a perfect square.*

Using computer we constructed the list of integral complete 4-partite graphs $K_{m,m,p,p}$ for $1 \le m \le 10000$ and $1 \le p \le 10000$. The list contains 57988 integral complete 4-partite graphs. Some of them are in Table 13.

m	p	zeros of the divisor			
1	1	-1	-1	-1	3
1	6	-1	-6	-2	9
2	5	-2	-5	-3	10
2	35	-2	-35	-5	42
3	10	-3	-10	-5	18
3	35	-3	-35	-7	45
3	88	-3	-88	-8	99
4	39	-4	-39	-9	52
4	165	-4	-165	-11	180
5	2	-5	-2	-3	10
5	21	-5	-21	-9	35
5	44	-5	-44	-11	60
5	68	-5	-68	-12	85
5	117	-5	-117	-13	135
5	266	-5	-266	-14	285
6	1	-6	-1	-2	9
6	391	-6	-391	-17	414

Table 13

4. Conclusion

In the paper we give sufficient conditions for complete 3-partite graphs $K_{m,m,p}$ and complete 4-partite graphs $K_{m,m,m,p}$ to be integral.

Corollary 11 gives sufficient conditions for a complete 3-partite graph $K_{m,m,p}$ to be integral. Are there complete 3-partite graphs $K_{m,m,p}$ that cannot be constructed by these conditions? Are these conditions necessary for a complete 3-partite graph $K_{m,m,p}$ to be integral?

Corollary 22 gives sufficient conditions for a complete 4-partite graph $K_{m,m,m,p}$ to be integral. Are there complete 4-partite graphs $K_{m,m,m,p}$ that cannot be constructed by these conditions? Are these conditions necessary for a complete 4-partite graph $K_{m,m,m,p}$ to be integral?

The method used in this paper can be used to explore integral complete r-partite graphs with $r-1$ partites of m vertices and one partite of p vertices, as the divisor of the graph can be devided by $(x-m)^{r-2}$, which leads to a quadratic equation $x^2-(r-2)mx-(r-1)mp=0$.

References

[1] BALIŇSKA, K., CVETKOVIČ, D., RADOSAVLJEVIČ, Z., SIMIČ, S., STEVANOVIČ, D. *A Survey on Integral Graphs*. Univ. Beograd. Publ. Elektrotechn. Fak. Ser. Mat 13 (2002) 42-65.
[2] CVETKOVIČ, D.M., DOOB, M., SACHS, H. *Spectra of graphs – Theory and Application.* Deutscher Verlag der Wissenschaften – Academic Press, Berlin – New York, 1980; 2nd Edition, 1982; 3nd Edition, Johann Ambrosius Barth Verlag, 1995.
[3] CVETKOVIČ, D.M., DOOB, M., GUTMAN, I., TORGAŠEV, A. *Recent results in the Theory of Graph Spectra.* North- Holland, Amsterdam, 1988.
[4] HARARY, F., SCHWENK, A. J. *Which graphs have integral spectra?* In: Graphs and Combinatorics. Lecture Notes in Math. 406, Springer-Verlag, Berlin, 1974, pp. 45-51.
[5] HÍC, P., NEDELA, R. *Balanced integral trees*. Math. Slovaca, 48 (1998), No. 5, pp. 429 - 445.
[6] HÍC, P., NEDELA, R., PAVLÍKOVÁ, S. *Front-divisors of trees*. Acta Math. Univ. Comeniane, vol. LXI, 1, 1992, p. 69-84.
[7] HÍC, P., POKORNÝ, M. *A note on integral complete 4-partite graphs*. Discrete Math. 308, pp. 3704-3705 (2008)
[8] HÍC, P., POKORNÝ, M., ČERNEK, P. *New sufficient conditions for integral complete 3-partite graphs.* DAppl. Anal. Discrete Math, 2, (2008) 276-284.
[9] HÍC, P., POKORNÝ, M. *On integral complete r-partite graphs*. Journal of the Applied Mathematics, Statistics and Informatics (JAMSI), V (2009), (in print)
[10] ROITMAN, M. *An infinite family of integral graphs*. Discrete Math. 52 (1984), no.2-3, 313-315.
[11] WANG, L.G. *A survey of results on integral trees and integral graphs*. Memorandum No. 1763, Twente, pp. 1-22, 2005, ISSN 0169-2690.
[12] WANG, L.G., LI, X., HOEDE, C. *Integral complete r-partite graphs*. Discrete Math. 283, (2004), no. 1-3, 231-241.
[13] WANG, L.G., WANG, Q. *Integral Complete Multipartite Graphs.* Discrete Math. 308 (2008), 3860-3870.
[14] WANG, L.G., LIU, X. *Integral Complete Multipartite Graphs* $K_{a_1\cdot p_1,a_2\cdot p_2,\ldots,a_s\cdot p_s}$ *with* $s=5,6$. Discrete Math. (2009) doi:10.1016/j.disc.2009.09.016