UNIVERSALITY OF LORENZ SYSTEM OF EQUATIONS

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Abstract. In our paper we demonstrated that famous Lorenz system of evolutionary equations describing the dynamics of climate systems is applicable in various qualitatively different fields, e. g. in hydrodynamics, climatology, optics, astrophysics, transport phenomena, biology and also in economical systems. It seems therefore, that this system of equations represents some type of universality, especially in the relation to the generation of a chaotic dynamics.

Key words: deterministic chaos, Lorenz equations, structuring of the universe, transport phenomena, economical systems

1.Introduction

The main aim of E. N. Lorenz was the solving of the problem of a long-time prediction of the weather. He derived three evolution equations (Lorentz system of equations LSE) [1]

$$\dot{q}_1 = \sigma q_2 - \sigma q_1 \quad (1)$$

$$\dot{q}_2 = -q_1 q_3 + r q_1 - q_2 \quad (2)$$

$$\dot{q}_3 = q_1 q_2 - b q_3, \quad (3)$$

where q_1 is one Fourier component of the velocity, q_2 and q_3 are components of the temperature field, σ is the Prandtl number, *r* is a constant defined by Rayleigh number and parameter *b* is defined by the relation $b = 1/(4 + \kappa^2)$, where κ is the (dimensionless) wave number corresponding to the selected mode.

As it is known Lorenz showed that the long-time forecasting of the weather is not possible. The reason is clear - a climatic system described by Eq. (1 - 3) exhibits a chaotic dynamics. This result was found by a serious mathematica1 analysis of LSE (for a good review see e. g. [2]). Arising of a regime of deterministic chaos in a system described by Eq. (1 - 3) is caused by an enormous sensitivity of the system on the initial conditions which results in appearance of a strange attractor - typical sign of systems working in the regime of a deterministic chaos.

Three fundamental characteristics of systems described by LSE (1 - 3) can be deduced:

1. In such systems there must be present a mechanism which guarantee the transition into the stationary state with the rate proportional to the value of appropriate variable (the presence of the term $\dot{q} = const.q$).
2. In the system there must exist the catalysis (the presence of the term $\dot{q}_i = const.q_j, i \neq j$).
3. Sub systems must interact nonlinearly (the presence of the term $\dot{q}_i = const.q_j q_k, i \neq k$).

It is evident that these conditions can occur in various systems, therefore it can be acceptable that LSE can be applicable in anorganic as well as in living systems. LSE was derived from basic hydrodynamical equations which can be interpreted as a proof that the dynamics described by LSE is valid in hydrodynamics.

Laser is a basic element of the nonlinear optics. Its theory was e1aborated by H. Haken. The same author proved that the fundamental equations determining the dynamics of the laser can be transformed into LSE [3]. The regime of deterministic chaos in the laser was really observed (see e.g. a review article [4]).

2.The structuralisation of the universe

Famous analysis of the problem of the structuralisation of the universe (J. Jeans [4]) showed the existence of the bifurcation point in which the originally homogenous material structuralised into equal c1usters with characteristic Jeans critical mass (the galaxies). The observations show that the masses of generated clusters exhibit a large spectrum of values. This is the clear evidence that the chaotic dynamics take part in the processes of the structuralisation of the universe.

In our analysis of these processes we use three equations (the continuity equation, the Euler's and Poisson's equations) in the form [5 - 6]

$$\frac{\partial \rho}{\partial t} = -3H\rho - \frac{1}{a}\nabla \cdot (\rho \mathbf{v}) \qquad (4)$$

$$\frac{d\mathbf{v}}{dt} = -H\mathbf{v} + \frac{1}{a}\mathbf{E} \qquad (5)$$

$$\nabla.\mathbf{E} = -4\pi G a^2 (\rho - <\rho>)\text{'} \qquad (6)$$

where ρ is the density, $<\rho>$ is the mean density of the expanding universe, H is the Hubble constant, $\mathbf{v}$ is the velocity, ∇ is the Hamilton's operator, G is the gravitational constant, $\mathbf{E}$ is the intensity of the gravitational field and a is a scale factor. For a flat and expanding universe the following relations are valid

$$a(t) = t^{2/3} \text{ and } < p > = const.a^{-3} = const.t^{-2} \qquad (7) \text{ and } (8)$$

Instead of definition $\mathbf{E} = \mathbf{E}_\rho - \mathbf{E}_{<\rho>}$ in Eq. (6) we introduce $\mathbf{E} = \mathbf{E}_{<\rho>} - \mathbf{E}_\rho$. The total derivative in Eq. (5) can be transformed using relationship $d/dt = \partial/\partial t + \mathbf{v}.\nabla$.

We will further deal with one dimensional universe to get only three evolution equation and to simplify the derivation. Regarding equations (4) and (6-8) and using approximation $p - < p > \cong qx(p - < p >)/a$, we obtain evolution equation for the intensity of the gravitational field E. Then we get three evolution equations

$$\dot{\rho} = -3H\rho - \frac{1}{a}\frac{\partial}{\partial x}(\rho v), \qquad (9)$$

$$\dot{v} = -Hv + \frac{1}{a}E - v\frac{\partial v}{\partial x}, \qquad (10)$$

$$\dot{E} = \left[\frac{4}{3}a^{-3/2} - 3H\right]x\frac{\partial E}{\partial x} - 4\pi G a \rho v + 4\pi G a \tilde{\rho} v. \qquad (11)$$

It was shown [see e. g. 7] that at the beginning of these processes the small changes of the variable can be expressed by an exponential function

$$\rho \approx \rho_0 \, exp(\gamma x) \tag{12}$$

The same can be supposed also for variables v and E. This supposition simplifies the spatial derivatives e. g. $\nabla\rho \approx \gamma\rho$. Under these conditions the investigated system of equations can be rewritten in the form

$$\dot{v} = a_0 E - a_1 v \tag{13}$$

$$\dot{E} = b_0 \rho v - b_1 v - b_2 E \tag{14}$$

$$\dot{\rho} = -c_0 v E - c_1 \rho - c_2 v, \tag{15}$$

where $a_0 = a^{-1}$, $a_1 = H$, $b_0 = 4\pi G a$, $b_1 = b_0 < \rho >$, $b_2 = \beta x(3H - 4a^{-3/2}/3)$,
$c_0 = \beta(\alpha + \beta)/4\pi G a^3$, $c_1 = 3H$, and $c_2 = \alpha < \rho > a^{-1}$.

It is seen that Eq. (13 - 14) are structurally identical with the first and the second of LSE. A simple analysis shows that the presence of the last term in Eq. (15) only shifts fixed points a little and the sign of the first term on the right side can be negative when both conditions $\alpha\beta > 0$ and $|\alpha\beta| > \beta^2$ are fulfilled. The local velocity v decreases in the direction of radius vector **r**, therefore the sign of the coefficient α is always negative. So we can conclude that the regime of the deterministic chaos can really occur in the process of the structura1isation of the universe.

3. Transport phenomena in solid materials

It is well known that many transport phenomena in solid materials need for their description three or more evolution equations, therefore the problem of the occurence of a chaotic dynamics in this case is very topicaI. The photoelectric transport in inhomogenous materials can be investigated as an example of such a phenomenon.

Let us suppose that the concentration of donor centres n_0 is a function of space coordinates. Under this condition the transport is described by three equations

$$\frac{\partial n}{\partial t} = A - r n n_0 - \nabla(n\mathbf{v} - D\nabla n), \tag{16}$$

$$\frac{d\mathbf{v}}{dt} = -\gamma_e \mathbf{v} - \frac{e}{m}\mathbf{E}, \tag{17}$$

$$\nabla\mathbf{E} = \frac{e}{\varepsilon}(n - n_0), \tag{18}$$

where n is the concentration of electrons, r is the coefficient of the recombination, A is the coefficient of the photogeneration, D is the diffusion coefficient, $\gamma_\varepsilon = 1/\tau$, where τ is the relaxation time, **E** is the intensity of the electric field, ε is the dielectric constant, **v** is the velocity of the electrons and m is their effective mass.

To get the usual form of the evolution equation from Eq. (17) we use the same operation as in the case of Eq. (5). Subtracting the relation $\partial n_0/\partial t = A - r n n_0$ from the Eq. (18) we can obtain the evolution equations in the form

$$\dot{n} = A - r n n_0 - \nabla \cdot (n\mathbf{v} - D\nabla n) \tag{19}$$

$$\dot{\mathbf{v}} = -\frac{e}{m}\mathbf{E} - \gamma_e \mathbf{v} - (\mathbf{v}.\nabla)\mathbf{v} \tag{20}$$

$$\dot{\mathbf{E}} = \frac{e}{\varepsilon}(n\mathbf{v} - D\nabla n) \tag{21}$$

The electric current usually flows in one direction, therefore one can transcribe Eq. (19 - 21) in the one-dimensional form. After some mathematical steps [details in reference 8] we use the same supposition as in the previous part ($\partial n/\partial x \approx \gamma\, n$ etc.) and the Eq. (19 - 21) transforms into the form

$$\dot{v} = a_0 - a_1 v\,, \tag{22}$$

$$\dot{E} = -b_0 n v + b_1 v - b_2 E\,, \tag{23}$$

$$n^* = c_0 v E - c_1 n^*\,, \tag{24}$$

where $a_0 = e/m$, $a_1 = \gamma$, $b_0 = e/\varepsilon$, $b_1 = -eA/\varepsilon b_2$, $b_2 = r < n_0 > -\gamma^2 D$, $< n_0 >$, being a mean value of the donor concentration, $n^* = n_0 - A/b_2$, $c_0 = \varepsilon(\alpha + \beta)/e$, and $c_0 = b_2$.

It is seen that there is a very good agreement between Eq. (1 -3) and Eq. (22 -24). Moreover we can demonstrate that when we introduce a renormalization into Eq. (22 - 24) the coefficients a_0 and a_1 are the same [8]. On the other hand numerical values of the coefficients present in Eq. (22 - 24) differs in general from those present in Eq. (1 - 3), therefore the chaotic regime may not arise in the photoelectric phenomena.

4.Economic systems

It is known that economic systems exhibit some periodical processes. The known simple models [see papers 9 - 10] of these processes give practically harmonic functions as a solution but the reality is different. A significant scattering in periods and amplitudes is observed, which implicates that some kind of chaotic dynamics take part in evolution of economic systems. According to several authors this observations can be understood as a result of the influence of some economic fluctuations which are supposed to be sinus profiled [10 and many others]. However this supposition is not realistic and moreover, small fluctuation are not able to explain the observed interval of scattering.

On a simple example we shall prove that the chaos can really appear in economic systems. R. M. Goodwin analyzed the interplay between two group s of investors [9]: N_1 prefer the expanded investments and the other N_2 the rationalized ones. Evolution equations are formulated for variables $x = (N_1 - N_2)/N$, $N = N_1 + N_2$ and a (the so called alternator determining the economic climate) in the form

$$\dot{x} = A[\sinh(a + kx) - x\cosh(a + kx)]\,, \tag{25}$$

$$\dot{a} = -B[a_0 \sinh(\alpha x) + a\cosh(\alpha x)]\,, \tag{26}$$

where k is the coordinator, A, B, a_0 and a are constants. These two equations have periodical solution and this result is interpreted as an explanation of Schumpeter's clock.

The alternator and the coordinator in Eq. (25 - 26) are independent characteristics and it is more probable that they act multiplicatively therefore we shall change the argument of hyperbolic function as follows $a + kx \to ak + kx = k(a + x)$. We propose to change the Eq. (26) into the form

$$\dot{a} = -B[a_0 \sinh k(x - x_0) + a \cosh k(x - x_0)], \quad (27)$$

where x_0 is the new constant. We select the coordinator k as the third state variable and if we want to have evolution equations analogical to LSE it is necessary to formulate the third evolution equation as follows

$$\dot{k} = C[\sinh(\delta x) + k \cosh(\delta x)] , \quad (28)$$

where δ is a constant. If we suppose that the arguments of hyperbolic functions are smaller than 1 (the inequality $x < 1$ is always valid) then we can use the approximations $\sinh y \approx y$ and $\cosh y \approx 1$. Using these relations we get evolution equations

$$\dot{x} = A[ka - (1 - k)x],$$
$$\dot{a} = B[k(x_0 - x) - a],$$
$$\dot{k} = C[\delta a - k] \cdot$$

It is seen that these equations are structurally identical with LSE except the term $(1 - k)$. However the presence of this term in Eq. (31) cannot destroy the occurence of the chaos if necessary conditions are fulfilled. Some other information can be found in the paper [11].

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