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REMARKS ON THE MEASURE DENSITY AND THE MAPPINGS ON THE SET OF POSITIVE INTEGERS

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SUMMARY.*In the first part we study the mappings which p preserve zero asymptotic density and we give a characterization of the sets of zero asymptotic density in the terms of bijections. The object of observations in the second and third part is the uniform density*

Let $\mathbb{N}$ be the set of natural numbers. For any subset $A \subseteq \mathbb{N}$ and $x > 0$, let $A(x)$ be the cardinality of $A \cap [0, x)$. The value $\limsup_x x^{-1}A(x) := \overline{d}(A)$ is called *the upper asymptotic density* of A, the value $\liminf_x x^{-1}A(x) := \overline{d}(A)$ is called *the lower asymptotic density* of A. If $\overline{d}(A) = \underline{d}(A)$ then we say that A has an *asymptotic density* and the value $\overline{d}(A) = \underline{d}(A)$ is called the *asymptotic density of the set* A. It is easy to see that this if and only if the limit $\lim_x x^{-1}A(x) := d(A)(= \overline{d}(A) = \underline{d}(A))$ exists. For more details on the asymptotic density we refer to the paper [G].

Lemma 1. *Suppose that $\mathbb{A} \subset \mathbb{N}$ is an infinite and $f : \mathbb{A} \to \mathbb{N}$ is such a mapping that*

$$\liminf_{\mathbb{A}} \frac{f(n)}{n} > 0. \tag{1}$$

Then for every $S \subset \mathbb{A}$ it holds

$$d(S) = 0 \Rightarrow d(f(S)) = 0. \tag{2}$$

Proof:. The inequality (1) implies that for some $\alpha > 0$ we have $n \cdot \alpha < f(n), n \in \mathbb{A}$. This implies that for $x > 0$ we have $f(n) \leq x$ yields $n \cdot \alpha < x$. Thus for $S \subset \mathbb{A}$ we get $f(S)(x) \leq S(\frac{x}{\alpha})$. From this we immediately obtain (2). $\square$

If $f : \mathbb{N} \to \mathbb{N}$ fulfills the condition (2) for every set $S \subset \mathbb{N}$ we say that f *preserves the zero density.*

For every set $S \subset \mathbb{N}$ it holds that $d(S) = 0$ if and only if $d(\mathbb{N} \setminus S) = 1$. From this we obtain immediately :

Lemma 2. *Let $f : \mathbb{N} \to \mathbb{N}$ be a permutation. Then f preserves the zero density if and only if for every $R \subset \mathbb{N}$ it holds*

$$d(R) = 1 \Rightarrow d(f(R)) = 1. \tag{3}$$

Theorem 1. *Let $g : \mathbb{N} \to \mathbb{N}$ be such a permutation that there exists a set $\mathbb{A} \subset \mathbb{N}$, $d(\mathbb{A}) = 1 = d(g(\mathbb{A}))$, and for every infinite $S \subset \mathbb{A}$, $d(S) = 0$, we have*

$$\liminf_{S} \frac{g(n)}{n} > 0. \tag{4}$$

Then g preserves the zero density.

Proof. Let $R \subset \mathbb{N}$ and $d(R) = 1$. Then $d(R \cap \mathbb{A}) = 1$. Thus $d(\mathbb{N} \setminus R \cap \mathbb{A}) = 0$. From (4) and Lemma 1 we get $d(g(\mathbb{N} \setminus R \cap \mathbb{A})) = 0$. This yields $d(g(R \cap \mathbb{A})) = 1$, thus $d(g(R)) = 1$. The assertion follows from Lemma 2. □

Example. *Let $\mathbb{N} \setminus \{n^2, n \in \mathbb{N}\} = A \cup B$ and $\mathbb{N} \setminus \{n^3, n \in \mathbb{N}\} = C \cup D$, where $A = \{a_1 < a_2 < ...\}, B = \{b_1 < b_2 < ...\}, C = \{c_1 < c_2 < ...\}, D = \{d_1 < d_2 < ...\}$. Moreover $A \cap B = \emptyset = C \cap D$. Let us consider the permutation $g : \mathbb{N} \to \mathbb{N}$ where $g(n^2) = n^3, n \in \mathbb{N}$ and $g(a_k) = c_k, g(b_k) = d_k$. If we suppose that the sets A, B, C, D have positive asymptotic density, then g fulfills the assumption of Theorem 1. If $d(A) \neq d(C)$ then g preserves the zero density but does not preserve the asymptotic density.*

Theorem 2. *Let $g : \mathbb{N} \to \mathbb{N}$ be an injective mapping and $\mathbb{A} \subset \mathbb{N}$, $\mathbb{A} = \{a_1 < a_2 < ....\}$ an infinite set.*
a) If

$$\lim_{n\to\infty} \frac{1}{a_n} \max\{g(a_j), j = 1, ..., n\} = 0 \tag{5}$$

then $d(A) = 0$.
b) If

$$\lim_{n\to\infty} \frac{g(a_n)}{a_n} = 0 \tag{6}$$

then $d(A) = 0$.

Proof. a) The values $g(a_j), j = 1, ..., n$ are different positive integers and so theirs maximum must be greater than $n - 1$. This implies

$$\frac{n}{a_n} \leq \frac{1}{a_n} \max\{g(a_j), j = 1, ..., n\}.$$

Now (5) implies $d(A) = 0$.
b)Put a_{k_n} such that $g(a_{k_n}) = \max\{g(a_j), j = 1, ..., n\}, n = 1, 2, ...$ Then

$$\frac{g(a_{k_n})}{a_n} \leq \frac{g(a_{k_n})}{a_{k_n}}$$

because $a_{k_n} \leq a_n$. The set $\{g(a_n), n = 1, 2, ...\}$ infinite and so $k_n \to \infty$ as $n \to \infty$. Therefore (6) implies (5). □

Aa a corollary of Theorem 2 we obtain the following characterization of the sets of zero density in the terms of permutations.

Corollary. *Let $\mathbb{A} \subset \mathbb{N}$, $\mathbb{A} = \{a_1 < a_2 < ....\}$ be an infinite set. Then $d(A) = 0$ if and only if there exists a permutation $g : \mathbb{N} \to \mathbb{N}$ fulfilling (6).*

Proof. The sufficiency follows from Theorem 2. If $d(A) = 0$ then $\frac{n}{a_n} \to 0$ for $n \to \infty$. Put $B = \mathbb{N} \setminus \mathbb{A} = \{b_n, n = 1, 2, ...\}$. The permutation g given by $g(a_n) = 2n, g(b_n) = 2n + 1$, fulfills (6). □

Uniform density

Let $x < y$ be two positive real number, put $A(x,y) := A(y) - A(x)$, thus this value gives us the number of elements of A between x, y.

Denote $\alpha^s(A) = \max_k A(k,k+s), \alpha_s(A) = \min_k A(k,k+s)$. It is well known that there exist the limits $\lim_s \frac{1}{s}\alpha^s(A) := \overline{u}(A)$ and $\lim_s \frac{1}{s}\alpha_s(A) := \underline{u}(A)$. The value $\overline{u}(A)$ is called the upper uniform density of A and the value $\underline{u}(A)$ is called the lower uniform density of A. The definition implies :
i)*If $A \subset \mathbb{N}$ and the set A contains the blocks of consecutive numbers of arbitrary length then $\overline{u}(A) = 1$.*
Let us denote $B = \mathbb{N} \setminus A$. Then $B(k,k+s) = s - A(k,k+s)$ thus $\underline{u}(B) = 1 - \overline{u}(A)$ and $\overline{u}(B) = 1 - \underline{u}(A)$. Therefore it holds
ii)*If $A \subset \mathbb{N}$ and the set $\mathbb{N} \setminus A$ contains the blocks of consecutive numbers of arbitrary length then $\underline{u}(A) = 0$.*

Theorem 1. *Let A, B be two infinite subsets of $\mathbb{N}$ such that A contains the blocks of consecutive elements from B of arbitrary length. Then $\overline{u}(A) \geq \underline{u}(B)$.*

Proof: The assumptions yield that for arbitrary n it is such k that $A(k,k+n) \geq B(k,k+n)$, thus $\max_k A(k,k+n) \geq \min_k B(k,k+n)$ and the assertion follows. □

If for $A \subset \mathbb{N}$ it holds $\underline{u}(A) = \overline{u}(A) := u(A)$ then we say that A has *uniform density*, and the value $u(A)$ is called *the uniform density* of A.

Let $A = \{a_1 < a_2 < ...\}$ be an infinite set. It is well known fact that if $\sum_n a_n^{-1} < \infty$ then A has the asymptotic density and $d(A) = 0$. Now we give an example that this does not hold for the uniform density. Consider the set $A = \cup_n \{n!+1, ..., n!+n\}$. From i) we see that $\overline{u}(A) = 1$ but it is easy to prove that in this case $\sum_n a_n^{-1} < \infty$.

Theorem 2. *Let $\{m_n\}$ be a sequence of positive integers, such that $(m_j, m_k) = 1$ for $k \neq j$. Put $A = \cup_{n=1}^{\infty} m_n\mathbb{N}$. Then*

(1) $\overline{u}(A) = 1$
(2) $\underline{u}(A) = 1 - \prod_{n=1}^{\infty}(1 - \frac{1}{m_n})$.

Proof: (1). The numbers $m_1, ..., m_n$ are relatively prime, thus due to the Chinese reminder theorem we obtain that there exists such a positive integer x_n that $x_n \equiv -j \pmod{m_j}$ for $j = 1, ..., n$. Therefore $x_n + j \in m_j\mathbb{N}$, j=1,...,n. This yields $x_n + 1, ..., x_n + n \in A$ and from i) we obtain $\overline{u}(A) = 1$.

(2).Put $A_n = \cup_{j=1}^{n} m_j\mathbb{N}$. Clearly $A_n \subset A$. It can be easily proved $u(A_n) = 1 - \prod_{j=1}^{n}(1 - \frac{1}{m_j})$ and so for $n \to \infty$ we obtain $1 - \prod_n^{\infty}(1 - \frac{1}{m_n}) \leq \underline{u}(A)$. Other inequality we obtain from the fact that $d(A) = 1 - \prod_{n=1}^{\infty}(1 - \frac{1}{m_n})$. □

Denote by Q_n, for $n = 2, 3, ...$ the set of positive integers which are not divisible by the $n-$ th power of prime number. Denote by $\mathcal{P}$ the set of all prime numbers. Then it holds $\mathbb{N} \setminus Q_n = \cup_{p \in \mathcal{P}} p^n\mathbb{N}$, where the union is considered through all prime numbers p. Thus $\underline{u}(\mathbb{N} \setminus Q_n) = 1 - \prod_{p \in \mathcal{P}}(1 - p^{-n}) > 0$ and so from ii) it follows that Q_n does not contains the blocks of consecutive integers of arbitrary length.

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Now we shall study one type of arithmetic functions from point of view of the uniform density of their range.

Lemma 1. *Let $f : \mathbb{N} \to \mathbb{N}$ be an arithmetic function fulfilling the condition*

(a) $\liminf_{n\to\infty} \frac{f(n+k)-f(k)}{n} > 0$ *uniformly for* $k = 1, 2, ...$

Then for every $A \subset \mathbb{N}$, $u(A) = 0$ *it holds* $u(f(A)) = 0$.

Proof: The condition (a) implies that for suitable $\beta > 0, n_0 \in \mathbb{N}$ we have

(1) $$f(n+k) - f(k) \geq \beta n,\ n \geq n_0,\ k = 1, 2, ...$$

Thus the set $F := f(\mathbb{N})$ can be represented in the form $F = F^{(1)} \cup \cdots \cup F^{(n_0)}$ where

$$F^{(i)} = \{f(i) < f(i+n_0) < ... < f(i+mn_0) < ...\},$$

for $i = 1, ..., n_0$. Let us denote $E^{(i)} = F^{(i)} \cap f(A)$. Thus $E^{(i)} = \{f(i+mn_0); i + mn_0 \in A, m \in \mathbb{N}\}, i = 1, ..., n_0$. Clearly $f(A) \subset E^{(1)} \cup ... \cup E^{(n_0)}$, therefore it suffices to prove $u(E^{(i)}) = 0, i = 1, ..., n_0$.

Let $k, n \in \mathbb{N}$ and

$$f(i+m_1n_0), ..., f(i+m_sn_0) \in [k, k+n]$$

for $m_1 < m_2 < ...m_s$, $m_j \in \mathbb{N}, i + m_jn_0 \in A, j = 1, ..., s$. Then

$$f(i+m_sn_0) - f(i+m_1n_0) \leq n.$$

From the other side the inequality (1) implies

$$f(i+m_sn_0) - f(i+m_1n_0) \geq \beta(m_s - m_1)n_0.$$

This yields $\beta(m_s - m_1)n_0 \leq n$ and so $m_s \leq m_1 + \frac{n}{\beta n_0}$. The numbers $i + m_jn_0, j = 1, ..., s$ belong to the interval $[r, r+\frac{n}{\beta}]$, where $r = i+m_1n_0$. We get $s \leq A(r, r+\frac{n}{\beta})$, in the other words

(2) $$E^{(i)}(k, k+n) \leq A(r, r+\frac{n}{\beta}),$$

thus $u(E^{(i)}) = 0$. □

Now we recall a well known property of uniform density. Denote for a prime number p and $A \subset \mathbb{N}$ by A_p the set of these elements of A which are divisible by p and not divisible by p^2.

In [P] it is proved the following statement: *Let* P *be such set of primes that* $\sum_P p^{-1} = \infty$. *Then for* $A \subset \mathbb{N}$ *it holds*

(3) $$(\forall p \in P; u(A_p) = 0) \Rightarrow u(A) = 0.$$

Lemma 2.*Let* P *be such set of primes that* $\sum_P p^{-1} = \infty$. *Denote for* $r = 1, 2, ...$ *by* $\mathbb{N}(r)$ *the set of all positive integers which have at most* r *distinct prime divisors from* P. *Then* $u(\mathbb{N}(r)) = 0, r = 1, 2, ...$

Proof: By induction with respect to r. Clearly $\mathbb{N}(0)_p = \emptyset$, for $p \in P$, thus (3) yields $u(\mathbb{N}(0)) = 0$.

It is easy to see that $\mathbb{N}(r+1)_p \subset p\mathbb{N}(r)$, thus from (3) we obtain $u(\mathbb{N}(r)) = 0 \Rightarrow u(\mathbb{N}(r+1)) = 0, r = 1, 2, ...$ □

Theorem 3. *Let* $f : \mathbb{N} \to \mathbb{N}$ *be an arithmetic function fulfilling the condition (a) from Lemma 1. Let* P *be such set of primes that* $\sum_P p^{-1} = \infty$. *Denote by* $\omega(n)$ *the number of distinct prime divisors from* P *of* $n, n \in \mathbb{N}$. *Let* f *fulfills moreover the condition*

(b) *There exists* $a \in \mathbb{N}, a > 1$ *that* $a^{g(\omega(n))} | f(n)$ *for* $n \in \mathbb{N}$. *Where* $g : \mathbb{N} \to \mathbb{N}$ *is such a function that* $g(n) \to \infty$ *for* $n \to \infty$.

Then $u(F) = 0$, *where* $F = \{f(n), n \in \mathbb{N}\}$.

Proof: Let $s \in \mathbb{N}$. The set F can be decomposed to $F = F_1 \cup F_2$, where $F_1 = \{f(j); j \in \mathbb{N}, a^s | f(j)\}$ and $F_2 = F \setminus F_1$. Clearly $\overline{u}(F_1) \leq a^{-s}$. We prove $u(F_2) = 0$. The condition (b) yields that there exists a nonnegative integer r that $F_2 \subset f(\mathbb{N}(r))$, where $\mathbb{N}(r)$ is the set from Lemma 2. Thus Lemma 1 implies $u(F_2) = 0$. Therefore $\overline{u}(F) \leq a^{-s}$ and for $s \to \infty$ we obtain $u(F) = 0$. □

We conclude this note by one sufficient condition under which an injective mapping preserves the uniform density.

Theorem 1. *Let* $g\colon \mathbb{N} \to \mathbb{N}$ *be an injection fulfilling the condition*

$$\lim_{n\to\infty} \frac{g(n+k)-g(k)}{n} = 1 \tag{1}$$

uniformly for $k = 1, 2, \cdots$. *Then* g *preserves the uniform density.*

For the proof we shall use the following statement proved in the paper [GLS].

Lemma. *Let* $S = \{s_1 < s_2 < ...\} \subset \mathbb{N}$ *be an infinite set. The* S *has the uniform density if and only if the fraction*

$$\frac{n}{s_{n+k} - s_k}$$

converges uniformly as $n \to \infty$, $k = 1, 2, ...$ *And in this case the value of its limit is equal to the uniform density of* S.

Proof of Theorem 1. The condition (1) yields that for two sequences $\{h_1(n,k)\}$, $\{h_2(n,k)\}$ such that $h_1(n,k) - h_2(n,k) \to \infty$, $n \to \infty$ uniformly for $k = 1, 2, \cdots$ we have

$$\frac{g\big(h_1(n,k)\big) - g\big(h_2(n,k)\big)}{h_1(n,k) - h_2(n,k)} \rightrightarrows 1, \quad n \to \infty \tag{2}$$

(As usually we use the symbol $\rightrightarrows$ for the uniform convergence.)
Let $A = \{a(1) < a(2) < \dots\}$ be an infinite set, which has the uniform density and $u(A) = \alpha$.

From Lemma we obtain

$$\frac{n}{a(n+k) - a(k)} \rightrightarrows \alpha, \quad n \to \infty \tag{3}$$

Put $g(A) = \{g\big(a(1)\big), g\big(a(2)\big), \dots\}$. These elements are not necessary arranged to their magnitude. Clearly $a(n+k) - a(k) \geq n$, and so $a(n+k) - a(k) \rightrightarrows \infty$ as $n \to \infty$. The relation (2) now implies

$$\frac{g\big(a(n+k)\big) - g\big(a(k)\big)}{a(n+k) - a(k)} \rightrightarrows 1, \quad n \to \infty \tag{4}$$

Therefore for suitable n_o the fraction on left side is positive for $k = 1, 2, \cdots$, thus $g\big(a(n_0+k)\big) > g\big(a(k)\big)$, $k = 1, 2, \cdots$. And so we see that the set $g(A)$ we can decompose into a union of disjoint sets

$$g(A) = B_1 \cup B_2 \cup \cdots \cup B_{n_0} \tag{5}$$

where

$B_j = \{g\big(a(j)\big) < g\big(a(j+n_0)\big) < \cdots g\big(a(j+rn_0)\cdots\} \; j = 1, \cdots, n_0.$

The relation (3) now implies

$$\frac{r \cdot n_0}{a\big(j+(r+k)n_0\big) - a(j+k\cdot n_0\big)} \rightrightarrows \alpha, \quad r \to \infty \tag{6}$$

Moreover the relation (2) yields

$$\frac{g\big(a\big(j+(k+r)n_0\big)\big) - g\big(a(j+k\cdot n_0)\big)}{a\big(j+(k+r)n_0\big) - a(j+k\cdot n_0\big)} \rightrightarrows 1, \quad r \to \infty \tag{7}$$

because the denominator is $\geq r \cdot n_0$ and so tends to ∞ uniformly for $k = 1, 2, \cdots$.

Thus from (6) and (7) we can deduce

$$\frac{r}{g\big(a\big(j+(k+r)n_0\big)\big) - g\big(a(j+k\cdot n_o)\big)} \rightrightarrows \frac{\alpha}{n_0}, \quad r \to \infty$$

and so $u(B_j) = \frac{\alpha}{n_0}$, $j = 1, \cdots, n_0$. From (5) we have $u\big(g(A)\big) = \alpha$. $\square$

Consider $g(n) = n + c \cdot \log n + O(1)$. Then $g(n+k) - g(k) = n + c \cdot \log\big(\frac{n}{k} + 1\big) + O(1)$, but $O \leq \log\big(\frac{n}{k} + 1\big) \leq \log(n+1)$ and g fulfills (1). Analogously it can be proved that

$$g(n) = n + c_1 \log_{r_1}^{n} + c_2 \log_{r_2}^{(n)} + \cdots + O(1)$$

where $r_1, r_2, \cdots, r_j > 1$ fulfills (1).

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