ABOUT AN EXPERIMENT WITH THE HARMONIC SERIES

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Abstract: This article describes a connection between the harmonic series and the Euler's number e.
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1. Introduction

In this contribution we are to describe a case when a computer was used for discovering a hypothesis in secondary school mathematics lessons. This hypothesis refers to the harmonic series

$$1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{n}+...$$

which is a well-known example of a divergent series (with sum $+\infty$) although it meets the necessary condition of convergence, i.e. a sequence of its n^{th} terms converges to zero. As you know, a sequence of partial sums s_n grows up very slowly.

$$s_n=1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{n}$$

In mathematical analysis textbooks (see e.g. [1]) we can learn that

$$s_{1000}=7.48...,\quad s_{1000000}=14.39,...\ .$$

In secondary school mathematics lessons we tried to demonstrate the growth of partial sums s_n with the help of computer technology.

2. Description of the experiment

If we wish to obtain values of s_n, we can use a calculator for first computations. Like that, we can elicit for instance (correct to seven decimal places):

s_1 = 1.0000000	s_6 = 2.4500000	s_{11} = 3.0198773
s_2 = 1.5000000	s_7 = 2.5928571	s_{12} = 3.1032107
s_3 = 1.8333333	s_8 = 2.7178571	s_{13} = 3.1801338
s_4 = 2.0833333	s_9 = 2.8289683	s_{14} = 3.2515623
s_5 = 2.2833333	s_{10} = 2.9289683	...

We were concerned by sums $s_1, s_4, s_{11}, \ldots$, where the partial sum reaches values of 1, 2, 3, . . . for the first time. The relevant indexes 1, 4, 11, … were marked as $p_1, p_2, p_3, \ldots$. So p_n is the index of such a partial sum of the harmonic series, for which the following is true:

$$s_{p_n-1} < n \ , \ \ s_{p_n} \geq n \ .$$

For getting other values of terms of the sequence $p_n {}_{n=1}^{\infty}$ the calculator is not longer sufficient because the sequence $p_n {}_{n=1}^{\infty}$ grows up very quickly, i.e. to get values of other terms it is necessary to add more and more terms of the harmonic series.

That’s why we have set a programme (it is a GW Basic version for computation of *M* terms)

```
10 REM Harmonic series
20 CLS : PRINT "Harmonic series" : PRINT
30 S#=0 : P#=1 : N=1 : M=12
40 S#=S#+1/P# : P#=P#+1 : IF S#<N THEN 40
50 PRINT N,P#-1,S#
60 N=N+1 : IF N<=M THEN 40
70 END
```

Calculations of numbers p_n with a growing *n* escalate their time demand; the time consumption is dependent also on the kind of a computer that is being used though. In the columns you can see the first 18 terms of the sequence $p_n {}_{n=1}^{\infty}$:

1	227	33 617	4 989 191
4	616	91 380	13 562 027
11	1674	248 397	36 865 412
31	4550	675 214	...
83	12367	835 421	

We have noticed that every term of this sequence is approximately triple to the term preceding. For this hypothesis verification we carried out calculations of $\frac{p_{n+1}}{p_n}$ (applying computer technology again) and we got the sequence (in columns again):

4	2.7136...	2.7182675...
2.75	2.7175...	2.7182862...
2.8181...	2.718040...	...
2.6774...	2.718021...	
2.7349...	2.7182825...	

At first sight it seems that its terms approach the Euler’s number *e*. Therefore it was possible to state the hypothesis

$$\lim_{n\to+\infty}\frac{p_{n+1}}{p_n}=e\,. \qquad (1)$$

Looking at it more closely though, this result is not so surprising, as we realize the connection between partial sums of the harmonic series and the integral

$$\int_1^A \frac{dx}{x}=\ln A\,.$$

If the hypothesis (1) is correct, it means that $p_n \,{}_{n=1}^{\infty}$ is a certain "quasi-geometric" sequence with the common ratio e. It is possible to assess the value of its other terms approximately, without any more computation of harmonic series partial sums.

It is really possible to show (see [2]) that

$$1-\frac{1}{p_n}<\ln\frac{p_{n+1}}{p_n}<1+\frac{1}{p_n} \qquad (2)$$

holds for every natural number *n*.

Because the sequence $\left\{\frac{1}{p_n}\right\}_{n=1}^{\infty}$ converges to zero, out of the three-sequence theorem it follows that

$$\lim_{n\to+\infty}\ln\frac{p_{n+1}}{p_n}=1$$

and the hypothesis (1) is therefore proved.

Now let's have a look at how to designate the other terms of the sequence $p_n \,{}_{n=1}^{\infty}$. Out of (2), through various conversions, we get

$$p_n e^{1-\frac{1}{p_n}}<p_{n+1}<p_n e^{1+\frac{1}{p_n}}\,, \qquad (3)$$

from which we can assess the term p_{n+1} once we know the term p_n.

Let's test this assessment at computing p_{10} by that of p_9. According to (3),

$$4550e^{1-\frac{1}{4550}}<p_{10}<4550e^{1+\frac{1}{4550}}$$

$$12365.4...<p_{10}<12370.9...$$

holds true and so $p_{10}\in\{12\,366, 12\,367, 12\,368, 12\,369, 12\,370\}$. We could see that the right value is $p_{10}=$ 12 367.

Let's consider the question of the p_{n+1} designation accuracy by applying the formula (3). We are interested in the $2\delta_n=\beta_n-\alpha_n$ length of the interval (α_n,β_n) where

$$\alpha_n=p_n\,e^{1-\frac{1}{p_n}}$$

$$\beta_n = p_n e^{1+\frac{1}{p_n}}.$$

Applying the l'Hospital's rule we come to

$$\lim_{n\to+\infty} 2\delta_n = 2e.$$

Consequently, if we set $p_{n+1} \approx \frac{1}{2}(\alpha_n + \beta_n)$, we depart from the real value by less than δ_n where δ_n converges to e.

3. Conclusion

In the process of the experiment we discovered with the students a nice connection between the harmonic series and the Euler's number e. This important constant was discovered in the process of the experiment in the natural way. This article, therefore, can be an appeal for discovering similar hypothesis of number series with the help of computer technology.

References

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