A NOTE ON INTEGRAL TREES OF DIAMETER 9

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Abstract. In [9] Híc and Pokorný described the method which helped them to find four integral trees of diameter 7. These trees were the first known integral trees of diameter 7. The method was based on joining the centres of two integral trees of diameter 3 by an edge. In this paper we explain that similar method could not be used to find an integral tree of diameter 9.

Key words: Integral Tree, Diameter, Divisor, Characteristic Polynomial.

1. Introduction

The notion of integral graphs was introduced by F. Harary and A. J. Schwenk in 1974 (see [4]). A graph G is called integral if all the zeros of the characteristic polynomial $P(G,x)$ are integers. Because the problem of characterizing integral graphs seems to be difficult, some authors restrict their investigations to trees. Results on integral trees of diameter k where $2 \leq k \leq 10$ can be found in [1, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15]. From these papers follows that integral trees of diameters 1, 2, 3, 4, 5, 6, 7, 8 and 10 can be constructed. But there is no integral tree of diameter 9. In [5] is proved that there are no balanced integral trees of diameter $4k+1$ where $k \geq 1$.

In [9] Híc and Pokorný described the method which helped them to find four integral trees of diameter 7. The method was based on joining the centres of two integral trees of diameter 3 by an edge. In this paper we explain that similar method could not be used to find an integral tree of diameter 9.

In [9] the authors code a balanced tree of diameter $2k$ $\mathrm{T}(n_k, n_{k-1}, \ldots, n_1)$, where n_j ($j=1, 2, \ldots, k$) denotes the number of successors of a vertex at the distance $k-j$ from the centre. The tree $\mathrm{T}(n_k, n_{k-1}, \ldots, n_1)\ \Theta\ \mathrm{T}(m_j, m_{j-1}, \ldots, m_1)$ is obtained by joining the centre w of $T(n_k, n_{k-1}, \ldots, n_1)$ and the centre v of $\mathrm{T}(m_j, m_{j-1}, \ldots, n_1)$ with a new edge. This tree is denoted by $\mathrm{T}(n_1, n_2, \ldots, n_{k-1}, n_k;1; m_j, m_{j-1}, \ldots, m_1)$.

In [5] the following Theorem is proved:

Theorem A (see [5; Theorem 2.1.]

Let D be a front-divisor of a tree T and C be the corresponding codivisor. Then

$$P(C;x) = \prod_{T_i \subseteq T-V(D)} P(T_i;x),$$

where T_i are connected components of T-$V(D)$ and in the case that D has the v_1-based loop, T_1 is the connected component of T-$V(D)$ with the u_1-based loop valued by -1. (u_1 is from the centre of T).

Notes:
This Theorem showed that the characteristic polynomial *P*(*C*;*x*) of a codivisor of a tree *T* can be expressed in terms of proper subtrees of T.
The properties of front-divisors and corresponding codivisors of trees and graphs are discussed in [2, 5, 7].

2. On Integral Trees of Diameter 9

Theorem 1.
Let *T*(*a*,*b*,*c*,*d*) Θ *T*(*e*,*f*,*g*,*h*) = *T*(*d*,*c*,*b*,*a*;1;*e*,*f*,*g*,*h*) be a tree of diameter 9. Then the tree *T*(*d*,*c*,*b*,*a*;1;*e*,*f*,*g*,*h*) is integral if and only if the following statements holds:
Corresponding divisor *D* (*d*,*c*,*b*,*a*;1;*e*,*f*,*g*,*h*) is integral.
The trees *T*(*b*,*c*,*d*) and *T*(*f*,*g*,*h*) are integral.

Proof.
It is easy to verify (see [2, 5 or 7]) that for a tree *T* = *T*(*d*,*c*,*b*,*a*;1;*e*,*f*,*g*,*h*) the corresponding front-divisor *D*(*d*,*c*,*b*,*a*;1;*e*,*f*,*g*,*h*) of *T* has characteristic polynomial

$$P\left(D\left(d,c,b,a;1;e,f,g,h\right);x\right)=x^{10}-\left(a+b+c+d+e+f+g+h+1\right)x^{8}+$$
$$\begin{pmatrix} bd+ad+d+de+df+dg+dh+ac+c+ce+cf+cg+ch+ \\ b+be+bf+bg+bh+ae+af+ag+ah+f+g+h+eg+eh+fh \end{pmatrix}x^{6}-$$
$$\begin{pmatrix} bd+bde+bdf+bdg+bdh+ade+adf+adg+adh+df+dg+dh \\ +deg+deh+dfh+ace+acf+acg+ach+cf+cg+ch+ceg+ceh \\ +cfh+bf+bg+bh+beg+beh+bfh+aeg+aeh+afh+fh \end{pmatrix}x^{4}+$$
$$\begin{pmatrix} bdf+bdg+bdh+bdeg+bdeh+bdfh+adeg+adeh+adfh+dfh+ \\ aceg+aceh+acfh+cfh+bfh \end{pmatrix}x^{2}-bdfh$$

Clearly, the connected components of *T*-*V*(*D*) have the form *T*(*b*,*c*,*d*), *T*(*f*,*g*,*h*), *T*(*c*,*d*), *T*(*g*,*h*), *T*(*d*), *T*(*h*), and isolated vertices. Because of *P*(*T*;*x*)=*P*(*D*;*x*).*P*(*C*;*x*), the tree *T* is integral if and only if the polynomials *P*(*D*;*x*) and *P*(*C*;*x*) have only integer zeros. □

We will show that there is no integral tree *T*(*d*,*c*,*b*,*a*;1;*e*,*f*,*g*,*h*) of diameter 9. The proof is based on the fact that
$P\left(D\left(d,c,b,a;1;e,f,g,h\right);0\right)=-bdfh<0$ and $P\left(D\left(d,c,b,a;1;e,f,g,h\right);1\right)>0$, which means that there is a zero of $P\left(D\left(d,c,b,a;1;e,f,g,h\right);x\right)$ in $\left(0;1\right)$.

c,*d*,*e*,*f*,*g*,*h* are larger than 3.

As $T(b,c,d)$ is integral, $P\left(D\left(b,c,d\right);x\right)=x^4-\left(b+c+d\right)x^2+bd$, and $P\left(D\left(b,c,d\right);0\right)=bd>0$, then $P\left(D\left(b,c,d\right);1\right)\geq 0\Rightarrow bd\geq b+c+d$.
Similarly, as $T(f,g,h)$ is integral, $P\left(D\left(f,g,h\right);x\right)=x^4-\left(f+g+h\right)x^2+fh$, and $P\left(D\left(f,g,h\right);0\right)=fh>0$, then $P\left(D\left(f,g,h\right);1\right)\geq 0\Rightarrow fh\geq f+g+h$.

Let us notice.

$$P\big(D(d,c,b,a;1;e,f,g,h);1\big)=1-a-b-c-d-e-f-g-h-1+$$
$$bd+ad+d+de+df+dg+dh+ac+c+ce+cf+cg+ch+$$
$$b+be+bf+bg+bh+ae+af+ag+ah+f+g+h+eg+eh+fh$$
$$-bd-bde-bdf-bdg-bdh-ade-adf-adg-adh-df-dg-dh$$
$$-deg-deh-dfh-ace-acf-acg-ach-cf-cg-ch-ceg-ceh$$
$$-cfh-bf-bg-bh-beg-beh-bfh-aeg-aeh-afh-fh$$
$$+bdf+bdg+bdh+bdeg+bdeh+bdfh+adeg+adeh+adfh+dfh+$$
$$aceg+aceh+acfh+cfh+bfh-bdfh$$

After a few simplifications we get

$$P\big(D(d,c,b,a;1;e,f,g,h);1\big)=-a-e+ad+de+ac+ce+be+ae+af+ag+ah+eg+eh$$
$$-bde-ade-adf-adg-adh-deg-deh-ace-acf-acg-ach-ceg-ceh-beg-beh$$
$$-aeg-aeh-afh+bdeg+bdeh+adeg+adeh+adfh+aceg+aceh+acfh$$
$$=(bdeh-beh-ceh-deh)+(acfh-acf-acg-ach)+(adfh-adf-afh+af)$$
$$+(bdeg-bde-beg+be)+(adeg-adg-deg)+(adeh-ade-adh+ad)$$
$$+(aceg-aeg-ceg+eg)+(aceh-ace-aeh+ae)+(ah-a)+(eh-e)+de+ac+ce+ag$$
$$=eh(bd-b-c-d)+ac(fh-f-g-h)+af(d-1)(h-1)+be(d-1)(g-1)$$
$$-dg+(a-1)d(e-1)g+ad(e-1)(h-1)+(a-1)(c-1)eg+a(c-1)e(h-1)$$
$$+a(h-1)+e(h-1)+de+ac+ce+ag$$
$$=eh(bd-b-c-d)+ac(fh-f-g-h)+af(d-1)(h-1)+(be-1)(d-1)(g-1)-d-g+1$$
$$+(a-1)d(e-1)g+ad(e-1)(h-1)+(a-1)(c-1)eg+a(c-1)e(h-1)$$
$$+a(h-1)+e(h-1)+de+ac+ce+ag$$
$$=eh(bd-b-c-d)+ac(fh-f-g-h)+af(d-1)(h-1)+(be-1)(d-1)(g-1)$$
$$+(a-1)d(e-1)g+ad(e-1)(h-1)+(a-1)(c-1)eg+a(c-1)e(h-1)$$
$$+a(h-1)+e(h-1)+(de-d)+ac+ce+(ag-g)+1>0$$

We proved that there is a zero of $P(D(d,c,b,a;1;e,f,g,h);x)$ in $(0;1)$, so the tree $T(d,c,b,a;1;e,f,g,h)$ could not be integral.

3. Conclusion

It is known that there are no integral balanced trees of diameter 9 (see [5]). In this paper we prove that there are no integral trees of the type $T(d,c,b,a;1;e,f,g,h)$. The problem of existence of integral trees of diameter 9 remains open.

References

[1] BALIŇSKA, K. CVETKOVIČ, D., RADOSAVLJEVIČ, Z., SIMIČ, S., STEVANOVIČ, D.: A Survey on Integral Graphs, Univ. Beograd. Publ. Elektrotechn.Fak. Ser. Mat 13 (2002) 42-65.

[2] CVETKOVIČ, D.M., DOOB, M., SACHS, H.: Spectra of graphs – Theory and Application, Deutscher Verlag der Wissenschaften – Academic Press, Berlin – New York, 1980; 2nd Edition, 1982; 3nd Edition, Johann Ambrosius Barth Verlag, 1995.

[3] HARARY, F.: *Graph Theory*, Addison – Wesley Publishing Company, Reading, Mass, 1969.

[4] HARARY, F. - SCHWENK, A. J.: Which graphs have integral spectra? In: *Graphs and Combinatorics. Lecture Notes in Math. 406*, Springer-Verlag, Berlin, 1974, pp. 45-51.

[5] HÍC, P., NEDELA, R.: Balanced integral trees. Math. Slovaca, 48 (1998), No. 5, pp. 429 - 445.

[6] HÍC, P., NEDELA, R.: Note on zeros of the characteristic polynomial of balanced trees, Acta Univ. M. Bellii, Math. no. 3 (1996), pp. 31 - 35.

[7] HÍC, P., PAVLÍKOVÁ, S.: Integral bibalanced trees, The collection of papers of the Faculty of materials science and technology, Trnava, vol. 3, 1995, p.227-233.

[8] HÍC, P., NEDELA, R., PAVLÍKOVÁ, S.: Front-divisors of trees, Acta Math. Univ. Comeniane, vol. LXI, 1, 1992, p. 69-84.

[9] HÍC, P., POKORNÝ, M.: There are Integral Trees of Diameter 7. Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. 18 (2006).

[10] HÍC, P., POKORNÝ, M.: On integral balanced rooted tree of diameter 10, Acta Univ. Mathaei Belii, Series Mathematics, no. 10 (2003), p.9-15. ISBN 80-8055- 812-4.

[11] POKORNÝ, M.: A Note on Integral Balanced Rooted Trees of Diameter 10. *Acta Fac. Paed. Univ. Tyrnaviensis,* Ser. C, 2004, no. 8. pp. 13-15.

[12] POKORNÝ, M.: Integral trees and Pell´s equations, *Acta Fac. Paed. Univ. Tyrnaviensis*, Ser. C, 2002, no. 6, p. 44-50.

[13] WANG L.G., LI X.L., ZHANG S.G.: Families of integral trees with diameter 4, 6, and 8, Discrete Applied Mathematics 136, 2004, 349-362.

[14] WANG L.G., LI X.L. : Integral trees with diameter 5 and 6, Discrete Mathematics 297 (1-3), 2005, 128-143.

[15] WANG L.G.: A survey of results on integral trees and integral graphs, Memorandum No. 1763, Twente, pp. 1-22, 2005, ISSN 0169-2690.